# Several Elementary Estimates of Prime Number Distribution and Their Improvements

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**Abstract**

This paper systematically reviews elementary estimation methods and their improved techniques for prime number distribution, focusing on the theoretical foundations and applied value of the Euler sieve, Chebyshev's theorem, and elementary proofs of the prime number theorem. It also analyzes the latest progress in improvements based on analytical methods and numerical optimization. By reviewing the mathematical principles and historical development of elementary methods, this paper illustrates their important role in revealing the laws governing prime number distribution, while also pointing out their limitations in accuracy and complexity. The article also speaks of possibilities of using contemporary elementary methods for building algorithms and cryptography and exploring the future of combining analysis tools and computational machinery. This article provides theoretical resources and practicable observations for future studies and interdisciplinary applications pertaining to prime number distribution.

**Keywords:** prime number distribution; elementary estimation; Euler sieve; Chebyshev's theorem; numerical optimization

## 1. Introduction

### 1.1 Background and Significance of the Prime Number Distribution Problem

Prime distribution problem has been one of the basic fields of number theory and seeks to study distribution laws and mathematical properties of prime numbers existing in the natural world. Euclid's proof of the infinitude of prime numbers from ancient Greece, this problem has stimulated systematic prime distribution study. Prime numbers as the kernel of integer factorizations are basic results of profound significance at the theoretical level, and also broadly applied to fields of cryptography, computer science, and information security. For example, encryption key from the RSA encryption technique relies on generation and distribution characteristic of large prime numbers [1]. The prime distribution problem has gradually changed from an elementary counting problem and became the kernel branch of modern number theory. The prime number theorem reveals the asymptotic relation of $\pi(x)$ and $x/\ln(x)$ and has been the foundation for describing prime distribution [2]. Elementary methods gained wide attention due to the fact that they avoid using complex analytic tools and are extremely valuable at education and theoretical proof levels. The elementary method provides an intuitive theoretical system with elementary arithmetic and combinatorial tools. Although though their precision and judging range are relatively small and limited, they provide the foundation base for further improving and developing. In recent years, following the development of calculating techniques and algorithms, prime distribution problem study has established an interdisciplinary bridge and stimulated interdisciplinary innovation [3]. The prime distribution problem has profound significance and far-reaching effect and influence not only because of its lovely mathematical characteristic, but also because it has great influence and effect on modern science and technique [3]. Elementary estimation method study provides an important vision for understanding prime number regularity and lays theoretical foundation for optimizing algorithms and exploring unknown tasks (such as the Riemann conjecture and others) [2].

### 1.2 Historical development of elementary methods

Histogenesis of elementary methods for the distribution of prime numbers can be traced back to the 19th century, where with elementary arithmetic and combinatorial tools, mathematicians sought to study the distribution of prime numbers instead of employing elaborate tools of complex analysis. Euler's proof of the infinity of prime numbers led the foundation stone to elementary method exploration and Gauss's conjecture of the prime counting function $\pi(x)\approx x/\ln(x)$ in 1792 provided an impetus to elementary exploration [4]. In the middle of the 19th century, Chebyshev made an effort to give an

elementary proof of the weak form of prime number theorem with elementary tools and obtained an upper and lower bound for π(x). This work provided an important breakthrough to elementary method exploration on prime distribution [2]. Chebyshev's proof employed no elaborate tools of complex analysis and relied on the analysis of the factorial and binomial coefficients and demonstrated the potential of elementary tools. Although only limited accuracy degree for his results could be achieved, he provided important inspiration to further exploration. In the early 20th century, Erdèsh and Selberg expanded elementary tools and succeeded in giving an elementary proof of the prime number theorem and avoiding using the zeta function of complex analysis [5]. This breakthrough demonstrated not only elementary tools' capability, but also provided necessary exploration for prime distribution further. In recent years, there has been an increasing amount of elementary tools' utilization to algorithm optimization and computational proof, for example, designing prime number screening and distribution statistics algorithms [3].

## 2. Classical elementary estimation methods

### 2.1 Euler sieve and prime number counting

Euler sieve is an ideal elementary approach to study the distribution of prime numbers. It has a central role in prime number counting due to its high efficiency and simplicity. The approach depends on Euler's elementary theorem of arithmetic, i.e., each positive integer can be represented as the distinct product of prime numbers, and an ascending sequence of prime numbers is obtained by repeatedly screening composite numbers. The Sieve of Euler, as compared with the Sieve of Eratosthenes, decreases repeated sieving immensely and increases efficiency significantly. The time complexity of the Sieve of Euler is O (n log n), and it can be used to count prime numbers over an extremely great range [6]. Its kernel is to accumulate the prime number table and the prime number count π(x) below an assigned value x, and it provides feasible data support to testify to the distribution law of prime numbers. The following formula describes the basic counting procedure and complexity analysis of the Sieve of Euler:

$$\pi(x) = \sum_{p \le x} 1 \tag{1}$$

$$\text{Time Complexity} = O(n \log\log n) \tag{2}$$

$$S(n) = \{k \in \mathbb{N} \mid k \le n, k \text{ is prime}\} \tag{3}$$

Formula 1 describes the prime number count function π(x), counting prime numbers up to x or smaller; Formula 2 describes the time complexity of the Euler sieve method, demonstrating high efficiency; Formula 3 establishes the prime number collection S(n) resulted from the sieve method. The above formulas reflect the mathematical basis of the Euler sieve method and its application to count [6]. The Euler sieve method cannot directly compute the asymptotic expression of π(x), but its results are an essential basis for prime number spacings and distribution statistics, and are widely used for prime number generation and testing in modern algorithm design [7]. The Euler sieve method has significant value in presenting an intuitive tool for primary features of prime number distribution. Based on mathematical properties of the sieve process, scholars can study the distribution pattern of prime numbers.

Chebyshev's Theorem

Chebyshev's Theorem is one of the high points in the history of distribution of primes. It gives elementary lower and upper bounds to the prime counting function π(x) and serves as the basis for developing the proof of the prime number theorem. In the mid-19th century, Chebyshev, applying the properties of factorials and binomial coefficients, proved the existence of constants c1 and c2 for which the following inequality holds good:

$$c \quad {}^{x} \quad \le \pi(x) \le c \quad {}^{x} \tag{4}$$

$${}^{1}\overline{\ln x} \qquad\qquad {}^{2}\overline{\ln x}$$

$$\theta(x) = \sum_{p \le x} \ln p \tag{5}$$

$$\psi(x) = \sum_{p^k \le x} \ln p \tag{6}$$

Formula 4 is the core of Chebyshev's theorem, which describes the upper and lower bounds of π(x) and verifies Gauss's conjecture that π(x) ≈ x/ln(x); Formula 5 defines the weighted prime sum θ(x), which represents the sum of the logarithms of prime numbers less than or equal to x; Formula 6 defines ψ(x), taking into account the contribution of prime powers[8]. These formulas characterize the growth trend of prime number distribution through elementary tools, avoid the complexity of complex analysis, and reflect the intuitiveness of the method [8]. Chebyshev's theorem provides quantitative bounds for subsequent research and inspires the elementary proofs of Erdesh and Selberg. The contribution of this theorem is that it introduces functions such as θ(x) and ψ(x), which provides a new perspective for studying the detailed structure of prime number distribution. Although its bounds are relatively loose, its elementary properties make it have important applications in numerical verification and algorithm design, such as optimizing computational efficiency in statistical analysis of prime number distribution [9]. The limitation of Chebyshev's theorem is that the constant range is large, but its method provides theoretical support for the improvement of elementary estimates.

### 2.3 Elementary Proof of the Prime Number Theorem

The prime number theorem, namely, π(x) ~ x/ln(x), is central to the study of prime number distribution. Its elementary proof marks the pinnacle of elementary methods. In 1949, Erdès and Selberg proved the theorem using a completely elementary method, avoiding reliance on complex analysis and the zeta function. Their proof is based on the following key formula:

$$\pi(x) \sim \frac{x}{\ln x} \tag{7}$$

$$\theta(x) \sim x \tag{8}$$

$$\psi(x) \sim x \tag{9}$$

Formula 7 is the asymptotic expression of the prime number theorem, indicating the equivalence of π(x) and x/ln(x); Formula 8 shows that the weighted prime sum θ(x) is asymptotically constant to x; and Formula 9 describes the asymptotic behavior of ψ(x), which together supports the derivation of the elementary proof [10]. These formulas derive the conclusion of the prime number theorem by analyzing the growth rate of the sum of prime numbers and combining complex combinatorial identities and inequality techniques, demonstrating the potential of elementary methods [10]. The breakthrough of this proof lies in its theoretical independence, which makes the verification of the prime number theorem no longer dependent on analytic number theory [11].

## 3. Improved Elementary Estimation Techniques

### 3.1 Improvements Based on Analytical Methods

Improvements of elementary estimation schemes are important milestones of prime distribution studies. Analytical-based refinements improve the estimation accuracy by using some tools of analysis while keeping elementary method simplicity intact. The classical elementary method, such as the Sieve of Euler and Chebyshev's Theorem, relies on pure arithmetic and combinatorial analysis and has limited accuracy. Analytical refinements, with tools like functions of logs and series analysis, give optimized estimation of the prime counting function π(x) independent of higher-order analysis. For example, with higher-order prime and θ(x) analysis, better upper and lower bounds can be constructed, giving tighter constant region for Chebyshev's Theorem. While keeping the elementary approach intact, the algebra adopts asymptotic analysis tools from analytic number theory and enhances the estimation accuracy appreciably. Further, there are also analytical-based refinements with extended prime spacing distribution studies enhancing asymptotic performance of the counting function through local prime density studies. The method allows increased theoretical descriptive accuracy and gives a new direction for prime distribution regularity verification. Improvements based on analytic schemes enhance theoretical

accuracy and also allow elementary scheme integration with state-of-the-art number theory. For example, with an added approximate expression for the logarithmic integral, the performance of $\pi(x)$ could be optimized and explained. This enhancement, with further intuitive simplicity of elementary approach, provides theoretical foundation for further optimizing algorithms and verifying numeric. Even though the method remains to be further improved and applied to address knotty problems, the value of the method cannot be discounted, and it lays the foundation for future studies on theoretical fronts.

### 3.2 Numerical Optimization and Algorithm Improvement

Numerical optimization and algorithm improvement are key modernizations of elementary estimation methods, aiming at improving efficiency and effectiveness of prime distribution estimation by computational means. The Euler sieve and the classic elementary methods take extended computation time to enumerate large prime numbers because of high computational complexity. Modern algorithms significantly reduce computational cost by optimizing the sieve process. For example, better algorithms for sieve reduce memory access and computation time through segmented processing and parallel computation and are feasible at the large scale for prime counting and generation. Moreover, numerical optimization of elementary technique-based estimation also covers approximate numerical computation of the prime counting function $\pi(x)$. Fast approximate computation methodologies of the logarithmic integral type, for example, fast algorithms for logarithmic integration, further improve estimation quality. Modernizations have demonstrated notable performance quality in practical applications, such as fast algorithms for large prime generation serving cryptography. Optimization of combinatorial identities of elementary estimation methods is also an important area of modern algorithm improvement. Modern algorithms are able to maintain elementary technique simplicity yet improve its ability of prime distribution pattern analysis by restructuring the mathematical design. For example, the better combinatorial sieve technique improves estimation of prime spacings by taking into consideration the prime number distribution's statistical properties.

### 3.3 Limitations of Modern Elementary Methods

Even though elementary estimation procedures themselves have markedly improved following refinements, elementary estimation shortcomings remain a significant problem for studies of prime number distribution. First of all, modern elementary method accuracy is limited by the limitations of the mathematical instruments used, and high accuracy like zeta-function-based analogues (analytical method) cannot be obtained. In this case, improved sieve and numerical algorithm procedures cannot detail $\pi(x)$ estimation precisely but cannot efficiently trace fine oscillations of prime number distributions, and this holds specifically for large prime gaps. In the second place, elementary method works complexity increases dramatically with problem size. In this case, ultra-large prime distributions cannot be considered because resource growth and complexity of algorithms used limit work opportunities. Further, theoretical basis of modern elementary method work continues to be inadequate for estimation of local prime distribution characteristics (for example, twin prime or prime distribution aggregates), and this hinders detailed mathematical modelling.

## 4. Conclusion

Fundamental estimation methods of prime number distribution and high-level improved methods are important branches of studies on number theory, providing transparent and efficient tools for discovering prime numbers' regularity on natural numbers. The paper revisits elementary classic methods such as Euler sieve, Chebyshev's theorem, and elementary proofs of prime number theorem and examines their theory significance and applied value. The methods reveal the asymptotic nature of prime counting function $\pi(x)$ with the aid of simple arithmetic and combinatorial tools and provide an important foundation for further studies. Optimization with the aid of numerical optimization tools and method-improved methods further promote the efficacy and exactness of elementary estimation methods and hold immense promise for contemporary algorithm design and cryptanalytic applications. Despite these improvement's immense enhancement on elementary method's practicability, they are still beset with restrictions on exactness and computational complexity as applied to complex prime distribution questions.

Prospects for developing elementary estimation methods include further integration of computational technique and analytic number theory to overcome these latter limitations of theoretical quality and range of usefulness. For example, an integration of recent numerical algorithms and analysis tools may produce essential breakthroughs in prime distance and local distribution properties. Once again, the elementary method's educational benefit of being intuitive increases its usefulness further and may be aided further by optimized learning structures. Additional prime distribution work will not only aid solutions of number theory's open questions such as the Riemann hypothesis, but also aid the interdisciplinary

extensions of prime numbers to security in information and computer science and offer new possibilities for integration of mathematics and technological advances.

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